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An Empirical Study on Effect of Estimation Risk on Portfolio Risk

[บทคัดย่อ]

การศึกษานี้มุ่งทำการศึกษาถึงผลกระทบของความเสี่ยงจากการประมาณค่าที่มีต่อความเสี่ยงของกลุ่มหลักทรัพย์ที่มีประสิทธิภาพอันยอมรับได้และกลุ่มหลักทรัพย์ที่ดีที่สุดโดยใช้วิธีการเบย์เชี่ยนภายใต้สมมติฐานที่หลักทรัพย์และกลุ่มหลักทรัพย์มีการกระจายแบบดิฟฟิวส์ (Diffuse) และแบบมีข้อมูลเบื้องต้น (Informative conjugate) กระบวนการศึกษานี้ได้ใช้ดัชนีรายอุตสาหกรรมของตลาดทุนในสหรัฐอเมริกา ผลจากการศึกษาขั้นตนบ่งชี้ว่าเมื่อได้รวมความเสี่ยงจากการประเมินค่าเข้าไปในกระบวนการจัดกลุ่มหลักทรัพย์แล้วกลุ่ม หลักทรัพย์ที่มีประสิทธิภาพอันยอมรับได้ที่เกิดขึ้นจากวิธีเบย์เชี่ยนไม่ได้แตกต่างจากกลุ่มหลักทรัพย์ที่มีประสิทธิภาพอันยอมรับได้ที่ได้จากวิธีการดั้งเดิม ผลสรุปสามประการจากการศึกษานี้คือ หนึ่งผลตอบแทนจากกลุ่มหลักทรัพย์ที่แท้จริงสามารถประมาณได้จากค่าถ่วงน้ำหนักของผลตอบแทนของตัวอย่างหลักทรัพย์ที่มีความถี่สูงและมีข้อมูลยาวนานเพียงพอ อย่างไรก็ตามตัวอย่างหลักทรัพย์ก็ไม่ได้เป็นตัวประมาณที่มีประสิทธิภาพ ผลสรุปประการที่สองคือเมื่อนำความเสี่ยงจากการประมาณค่ามาพิจารณาในการจัดกลุ่มหลักทรัพย์แล้ว ความเสี่ยงของกลุ่มหลักทรัพย์ที่ได้จะมีค่ามากกว่าความเสี่ยงของกลุ่มหลักทรัพย์แบบดั้งเดิมส่งผลให้กลุ่มหลักทรัพย์ที่ยอมรับได้ที่ได้จากวิธีดั้งเดิมและแบบเบย์เชี่ยนไม่แตกต่างกันแต่เส้นขอบเขตของกลุ่มหลักทรัพย์ที่มีประสิทธิภาพจะอยู่ด้านขวาของกลุ่มหลักทรัพย์ที่มีประสิทธิภาพดั้งเดิมเนื่องจากมีความเสี่ยงจากการประเมินค่าที่สูงขึ้น ผลสรุปประการที่สามได้จากผลการวิจัยเชิงประจักษ์โดยสามารถแบ่งออกได้สองผลสรุปดังนี้ การจัดกลุ่มหลักทรัพย์แบบดั้งเดิมที่ไม่ได้นำความเสี่ยงจากการประเมินค่าเข้ามาพิจารณาจะส่งผลให้กลุ่มหลักทรัพย์ที่มีประสิทธิภาพที่ได้มีประสิทธิภาพที่ด้อยกว่ากลุ่มหลักทรัพย์ที่มีประสิทธิภาพจากวิธีเบย์เชี่ยนเนื่องจากอรรถประโยชน์ที่ต่ำกว่าวิธีเบย์เชี่ยนโดยค่าความเสี่ยงของกลุ่มหลักทรัพย์จากวิธีเบย์เชี่ยมีค่าสูงกว่าความเสี่ยงของกลุ่มหลักทรัพย์แบบดั้งเดิมประมาณ 40 ถึง 80 เบสิสพ้อนท์ และ 100 ถึง 220 เบสิสพ้อนท์เมื่อตัวอย่างผลตอบแทนของหลักทรัพย์มาจากปบ บรายอาทิตย์ และรายเดือนตามลำดับ ผลสรุปประการที่สองจากการวิจัยเชิงประจักษ์คือผลตอบแทนเฉลี่ยส่วนเกินของกล่มหลักทรัพย์แบบเบย์เชี่ยนสูงกว่าผลตอบแทนเฉลี่ยส่วนเกินของกลุ่มหลักทรัพย์แบบดั้งเดิม,แบบที่มีน้ำหนักการลงทุนตามมูลค่าหลักทรัพย์และแบบที่มีน้ำหนักการลงทุนเท่ากันทุกหลักทรัพย์ที่ 36, 384, and 144 เบสิสพ้อนท์ตามลำดับ

[ABSTRACT]

THIS study explores effect of estimation risk on an admissible efficient set and an optimal portfolio based on a Bayesian framework assuming diffuse prior and informative conjugate prior distribution functions. Based on the U.S. sectorial index, the result indicated that, when estimation risk is taken into account, the admissible efficient set is not changed. Therefore, three conclusions can be drawn. First, true portfolio returns can be represented by weighted average sample returns given that samples are drawn from high frequency data with a long average period. However, historical sample average is not an efficient estimator for true parameters. Second, portfolio risk or variance, when estimation risk is built into a decision, is affected by a scale factor. Therefore, a Bayesian admissible efficient set will always lies to the right of the traditional admissible efficient set due to higher risk from estimation. Third, portfolio decisions based on a traditional approach, ignoring estimation risk, would lead to a suboptimal portfolio due to utility loss caused by underestimation of risk. Empirical results show that annualized Bayesian portfolio risk is larger than that of a traditional portfolio by approximately 40 to 80 basis points for a weekly index return interval and approximately 100 to 220 basis points for a monthly index return interval. Moreover, The annualized average excess portfolio return from Bayes-Stein shrinkage portfolio is higher than those of traditional, passive, and naïve portfolio by 36, 384, and 144 basis points, respectively.

1. Introduction

WHEN making a decision in portfolio selection under uncertainty, investors have long followed the practice of modern portfolio theory as documented by Markowitz (1952). Traditional portfolio allocation assumes known parameters with stationarity. In other words, traditional practice assumes that the joint probability density function of asset returns and true population mean vector and variance-covariance are known and parameters possess a stationary property. As a result of traditional assumptions, expected utility can be evaluated by substituting point estimates of sampling moments in the utility function. However, the joint probability density function of asset returns and parameters are usually not completely known. Therefore, in the portfolio selection process, we encounter not only the uncertainty of the future asset return generating process, but also the uncertainty of the functional form of the joint probability density function and of asset return parameters. These uncertainties are called estimation risks. The "estimation risk" comes from both choosing poor probability models and ignoring parameter uncertainty.

The common practice in portfolio selection for the traditional perspective is utilizing a single unknown parameter, such as assuming that the expected return for the portfolio is known but the volatility of asset returns is not known. To solve a portfolio selection problem is to find the appropriate weight of investment (asset allocation) in order to minimize return volatility given the expected return. By assuming one parameter is known, estimation risk is not treated properly. The contribution of this study is to provide empirical evidence of estimation uncertainty on the admissible efficient set based on the analytical works of Brown (1979) ,Bawa (1976), and Shrinkage estimator of Jorion (1986).

There are four pieces of related works discussed in this study. The first discussion is to explore effect of estimation risk on the admissble efficient set. The second aspect is to examine whether an optimal portfolio suggested by traditional approach and an optimal portfolio incorporating estimation risk are different or not. The third study is to analytically discuss loss in utility due to the effect of estimation risk in portfolio formation process. The last facet is to provide empirical evidence regarding effect of estimation risk on an optimal portfolio. This study explores the effect of estimation risk on an admissible efficient set and an optimal portfolio based on analysis under a Bayesian framework assuming a diffused prior density function and informative prior based on a selected conjugate density distribution.

Section Two discusses the evolutions of past studies regarding the estimation risk and the application of Bayesian Portfolios concept. Section Three explores the effects of estimation risk on portfolio risk and portfolio return and loss in utility. Data and Empirical evidence based on U.S. sectorial index returns adjusted for dividend are provided in Section Four and Five, respectively. The last section is the conclusion.

2. Estimation Risk and Bayesian Portfolio Selection

ONE of the fundamental propositions in modern finance theory is that security risk should be viewed in the context of a portfolio. Jorion (1986) stated that "...It is astonishing then that estimation techniques in finance have not recognized the implications of this result for efficient estimation of unknown parameters."

Using the classical mean-variance framework, where no attention is paid to uncertainty about the expected value and covariance matrix of asset returns, investors may underestimate portfolio risk and be willing to invest in a traditionally sub-optimal portfolio. Adler and Dumars (1983) documented that determining the optimal portfolio composition of the traditional approach is not correct because there is no statistical approach taking into account the estimation risk. Jorion (1985) explored estimation risk in an international portfolio context and found that estimation risk due to uncertain mean returns has a considerable impact on optimal portfolio selection. Britten-Jones (1999) also performed an empirical test of an international efficient portfolio by testing mean variance efficiency using the regression approach and found that sampling error in estimates of the weights of a global efficient portfolio is large. The result implies that there is no statistical support for portfolio diversification as suggested by the traditional portfolio approach.

Empirical evidence from Klein and Bawa (1976, 1977), Adler and Dumars (1983), Jorion (1985, 1986), Frost and Savarino (1986), Britten-Jones (1999), Polson and Tew (2000), and Greyserman et al. (2006) indicated that optimal members of traditional portfolios are different from those of portfolios incorporating estimation risk, and that it is more efficient to incorporate estimation risk in the portfolio selection process. Klein and Bawa (1977) also documented that risk-averse investors tend to invest relatively more in securities about which they have more information.

In portfolio selection within a Bayesian framework, optimal weights of investment are based on maximization of expected utility conditional on the predictive distribution of asset returns. Diffuse prior or non-informative prior distribution is widely used in previous work to alleviate the effect of estimation risk, such as by Klein and Bawa (1976), Bawa (1979), Brown (1979), etc. However, the estimation error is not reduced. Performance of the portfolio can be improved if the informative prior that reduces estimation risk is correctly specified.

Frost and Savarino (1986) suggested an informative prior distribution where all securities have identical expected returns, variances, and pairwise correlation coefficients. Such informative priors would reduce the estimation error because posterior estimates of parameters will be drawn from a specific distribution toward the average values of those parameters for all securities in the population or drawn toward the grand mean of those parameters. According to Barberis (2000), the empirical evidence indicates that investors with long horizon investments who ignore parameter uncertainty may over allocate to stocks by a sizeable amount. This suggests that estimation risk should be incorporated in portfolio selection decision.

2.1 Clarification of Variable and Notation of Portfolio Selection Process:

Let Rit = $[(r_{11}, r_{12}, ..., r_{1t})', (r_{21}, r_{22}, ..., r_{2t})', ..., (r_{n1}, r_{n2}, ..., r_{nt})']$ denote the random return vector representing rates of return on asset i, i = 1, 2,..., n, in period t, $\tilde{r}_{it}$ denote a vector or random variable, namely future security returns, w denote a vector of proportions of wealth invested in securities, $R(\tilde{r}, w)$ denote a random return vector resulting from an investor's decision and $f(R|\theta)$ and $p(R_t, \theta)$[1] denote a joint probability density function for random return observations R_{it} and a parameter vector θ. The parameter vector under portfolio allocation contains the true value of mean and covariance of asset returns, $\theta=(\mu, \Sigma)$. Furthermore, assume that data consists of a random sample return of T_i observations on each sectorial return.

Optimal investment decisions by any rational investor is determined from a golden axiom which complies with that of the Von Neumann-Morgenstern axiom, stating that an investor chooses an alternative investment that maximizes the expected utility of return on his/her investment. The problem of portfolio optimization is reduced to indicating the efficient frontier or the set of portfolios that have maximal expected portfolio returns given a specific level of expected portfolio variance or the set of

[1] The joint probability density function can be stated in the form of $f(R|\theta)$. The interpretation of $f(R|\theta)$ is the density function of security return given that the true population parameter is known or $f(\theta)$ is treated as a constant.

portfolios that have minimal expected portfolio variance given a specific level of expected portfolio return. Finally, let $U(\tilde{R})$ denote any investor's utility function defined over the random return vector, $\tilde{R}$. If the true parameter, θ, is known, an investor would choose an appropriate weight of investment that maximizes the expected utility as stated below:

$$\underset{\tilde{R}|\theta}{E}(U) \equiv \int_R U[R(\tilde{r}, w)] f(R|\theta) dR \quad (1)$$

In short, the optimal portfolio decision is to determine optimal weights allocated to each sector or to find the solution from a quadratic optimization that minimizes portfolio risk subjected to the constraint set. Solution of the optimization problem or weight invested in each sector under mean variance efficient portfolio, X_{EV}, is determined as:

$$X_{EV} = \frac{1}{\iota^T \Sigma^{-1} E(R_i)} \Sigma^{-1} E(R_i) \quad (2)$$

The optimal weights of a global minimum portfolio assigned to each sector, X_{GM}, are determined as:

$$X_{GM} = \frac{1}{\iota^T \Sigma^{-1} \iota} \Sigma^{-1} \iota \quad (3)$$

From the solution of global minimum optimal weight, the variance-covariance matrix is the key factor in the problem. This implies that estimation risk can be incorporated into the portfolio selection process by adjusting the variance-covariance matrix. Updating or adjusting the variance-covariance matrix can be done under the Bayesian framework. As suggested by Zellner and Chetty (1965), incorporating parameter uncertainty in any decision requires a derivation of predictive probability which can be done by integrating out the unknown parameter.

2.2 Effect of Estimation Risk on Portfolio Return and Portfolio Risk: An Analytical Analysis

In order to apply modern portfolio formation concepts, an investor must form expectations about the future performance of all securities in his/her universe. Future asset returns distribution on a set of n securities are assumed to be multivariate normal distributed with mean μ and covariance matrix Σ, where μ is an n x 1 vector, and Σ is an n x n positive definite symmetric matrix.

Let R represent the return vector with a dimension of t x 1 where t is the number of observations and W represents the proportion of investment vector with the dimension of n x 1. Sample mean and sample covariance can be defined as follows.

$$m_i = \frac{1}{t} \sum_{t=1}^{T} R_i \,, \; i = 1,2,\ldots,n \quad (4)$$

$$S = \frac{1}{t-1} \sum_{t=1}^{T} (R_i - m_i)'(R_j - m_j) \,, \; i,j = 1,2,\ldots,n \quad (5)$$

where : m = sample mean vector
S = sample variance-covariance matrix
t = number of observations
n = number of assets

Taking estimation risk into account, Kalymon (1971), Winkler (1973), Barry (1974) and Bawa, Brown, and Klein (1979) suggested a Bayesian framework under three states of prior knowledge. The first state assumes that the population or true parameters, μ and Σ, are known. The first state is typically assumed in traditional portfolio selection or mean variance analysis. The second state assumes that the true variance-covariance Σ is known and μ is not known. Kalymon (1971) suggested Bayesian portfolio selection under the second state. The third state assumes that both population parameters μ and Σ are not known. In this study, the analysis follows the third state as suggested by Winkler (1973).

From the concept of conditional probability, we can easily see that $f(R|\theta) = \frac{f(R,\theta)}{f(\theta)}$ .Let $\propto$ denote proportional density function. We can incorporate a constant into $\propto$ and rewrite $f(R|\theta)$ as $f(R|\theta) \propto f(R,\theta)$.

2.2.1 The First State: Both μ and Σ are Known

In this state, an investor estimates the population mean μ with sample mean m and population variance-covariance matrix Σ with sample variance-covariance matrix S. Portfolio return, $W'\mu$, is estimated by $W'm$ and the portfolio variance, $W'\Sigma W$, is estimated by $W'SW$. The result of the first state is equal to the result of the diffuse prior Bayesian portfolio as shown in Appendix A. When sample size is very large and approaches infinity, variance of portfolio equals $W'SW$.

2.2.2 The Second State: Σ is Known and μ is Not Known

Follow Kalymon (1971), in this state, an investor treats S as the "known" value of Σ and uses both prior and other information to estimate the unknown mean μ. The prior information used in this study is data base prior which is obtained from past data or the prior information is obtained from the asset return itself of the current return of n assets. A posterior mean, either diffuse or informative multivariate normal prior distribution, is the sample mean return m.

According to Kalymon (1971), the variance-covariance matrix is specified as $S + \frac{1}{T}S$ or $S(1+\frac{1}{T})^2$, where T represents number of observations. Portfolio return is $W'm$ and the portfolio variance is $W'[1+\frac{1}{T}]SW$. The portfolio variance can be rewritten in terms of the portfolio variance of the first state as follows:

$$[1+\frac{1}{T}]\, W'SW = [1+\frac{1}{T}] * \text{Portfolio variance of the first state} \quad (6)$$

2.2.3 The Third State: Both μ and Σ are Not Known

In this state, informative prior Bayesian with a specified prior conjugate distribution is applied. Suppose an investor assumes that joint distribution of $\tilde{\mu}$ and $\tilde{\Sigma}$ is normal-inverted Wishart family. For a multinormal process with unknown mean vector $\tilde{\mu}$ and unknown variance-covariance matrix $\tilde{\Sigma}$, the corresponding family of conjugate prior distributions is the normal-inverted Wishart family. Following Winkler (1973), the marginal distribution of $\tilde{\Sigma}$ is inverted Wishart. The distribution of mean return $\tilde{\mu}$ conditional on the variance-covariance matrix is normal with a mean vector of m and a covariance matrix of $\frac{1}{T}\Sigma$. Symbolically, this can be written $f_N(\mu \mid \Sigma) \sim N(m, \frac{1}{T}\Sigma)$.

The predictive distribution of is a multivariate t distribution with the following two moments:

$$E(R) = m = \text{sample mean} \quad (7.1)$$

$$\mathrm{Var}(\tilde{R}) = (\frac{d}{d-2})(\frac{T+1}{T})(\frac{T-1}{T})S = [\frac{(T^2-1)(T-n+1)}{T^2(T-n-1)}]S \quad (7.2)$$

Portfolio return is and the portfolio variance is

$$W'[\frac{(T^2-1)(T-n+1)}{T^2(T-n-1)}]SW = [\frac{(T^2-1)(T-n+1)}{T^2(T-n-1)}]\, W'SW \quad (8)$$

where: $[\frac{(T^2-1)(T-n+1)}{T^2(T-n-1)}]$ is a scalar term[3] larger than 1.

With the diffuse prior and selected conjugate prior distributions, we would benefit from the analytical Bayesian framework by having the closed form of the first two moments of asset return as shown in three states of analysis discussed above.

2 See the Appendices A and B for proofs.

3 For details see Appendix C

2.3 Bayesian Framework Treating all Assets are Identical: Bayes-Stein Shrinkage

Within this framework, the prior belief is that all assets will converge to a common mean or grand mean. True parameters can be estimated by assigning appropriate weights to the historical sample mean and the grand mean depending on prior information or knowledge of asset returns. If an investor has high confidence about the asset return distribution due to long historical data being used, less weight will be given to the grand mean. On the other hand, if not enough information regarding asset returns is on hand, the investor would assign more weight to the common mean. According to Jorion (1986), the selected informative conjugate prior on average return is given by the following.

$$p(\overline{R}|\eta,\lambda) \propto \exp[-(\frac{1}{2})(\overline{R}-\underline{1}\eta)'(\lambda\Sigma^{-1})(\overline{R}-\underline{1}\eta)] \quad (9)$$

where: R = historical or sample average return

η = grand mean

λ = prior precision

Given that the vector of observed returns on any assets follows the normal distribution which can be stated as

$$r_t \sim NID(\mu,\Sigma), \qquad t = 1,2,\ldots,T \quad (10)$$

Applying concept of the James-Stein shrinkage estimator, Jorion (1986) suggested the use of the selected informative prior as in Equation (9) and inferred that the predictive density function $p(\underline{r}|\overline{R},\Sigma,\lambda)$ is multivariate normal with mean and variance as stated below.

$$E[\underline{r}] = (1-\omega)\overline{R} + \omega\underline{1}\eta \quad (11)$$

$$V[\underline{r}] = \Sigma(1+\frac{1}{T+\lambda}) + \frac{\lambda}{T(T+1+\lambda)}\frac{\underline{1}\underline{1}'}{\underline{1}'\Sigma^{-1}\underline{1}} \quad (12)$$

where: ω = weight assigned to grand mean

$E[\underline{r}]$ = vector of future rate of return derived from the predictive density function

$$\omega \equiv \frac{\lambda}{T+\lambda} \quad (13)$$

1 = vector of unity

λ = precision parameter

Following Jorion (1986), the grand mean is treated as the average return for the global minimum portfolio. The grand mean can be calculated as the product of the global minimum weight and the historical average return on each asset.

$$\eta \equiv w'\overline{R} \equiv \frac{\underline{1}'\Sigma^{-1}}{\underline{1}'\Sigma^{-1}\underline{1}}\overline{R} \quad (14)$$

An empirical Bayesian approach lets the data speak through the precision parameter, λ. This means that λ is directly estimated from the data. Given that probability density function of the precision, $p(\lambda|\mu,\eta,\Sigma)$ is a gamma distribution with mean $\frac{(N+2)}{d}$, where d is defined as $(\overline{R} - \underline{1}\eta)'\Sigma^{-1}(\overline{R} - \underline{1}\eta)$. The shrinkage coefficient, ω, and precision, as constructed by Jorion (1986) are given below.

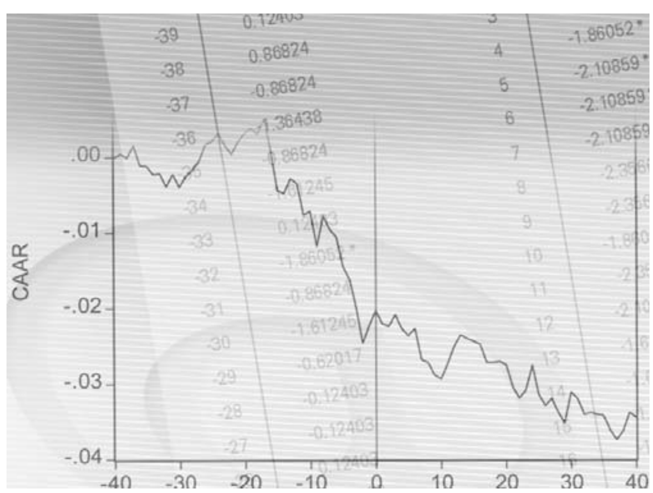
$$\hat{\omega} = \frac{N+2}{(N+2)+(\overline{R}-\eta\underline{1})'T\Sigma^{-1}(\overline{R}-\eta\underline{1})} \quad (15)$$

$$\lambda = \frac{(N+2)(T-1)}{[(\underline{R}-\eta\underline{1})'S^{-1}(\underline{R}-\eta\underline{1})(T-N-2)]} \quad (16)$$

Zellner and Chetty (1965) suggested a predict variance-covariance estimator when the variance-covariance parameter is not known, as shown below.

$$\hat{\Sigma} = \frac{T-1}{T-N-2}S \quad (17)$$

where: S = unbiased sample variance-covariance matrix



3. Effect of Estimation Risk on Efficient Set and Utility Loss

3.1 Effect of Estimation Risk on Efficient Set

Two points can be concluded from the analytical Bayesian portfolio of three states discussed in the last section. The first is the conclusion drawn for portfolio return and the second is the conclusion for portfolio risk or variance when estimation risk is taken into account. Portfolio returns are the same in all three states, $W'm$. Klein, Bawa, and Brown (1979) showed that in the limiting and diffuse prior case, the mean of the relevant predictive distribution of returns is given by the sample mean.

Portfolio risk or portfolio variance is different by a scale factor. This means that portfolio variance in all states has a common factor, $W'SW$. The variance-covariance matrix in the second and third states can be written in the form of the variance-covariance matrix of the first state. As shown in Table 1, as estimation risk is incorporated in the portfolio formation process, portfolio risk or variance of the portfolio in state 2 and state 3 is larger than that of the portfolio in state 1 by $[1+\frac{1}{T}]$ and $[\frac{(T^2-1)(T-n+1)}{T^2-(T-n+1)}]$, respectively.

Table1: This table shows portfolio return and portfolio variance under three states of analysis. The first state assumes that population or true parameters, μ and Σ, are known. The first state is typically assumed in traditional portfolio selection or mean variance analysis. The second state assumes that true variance-covariance Σ is known and μ is not known. The third state assumes that both population parameters μ and Σ are not known.

State of analysis	Portfolio Return	Portfolio Variance
Both μ and Σ are known	$W'm$ $W'SW$	
Σ is known and μ is not known	$W'm$	$[1+\frac{1}{T})$
Both μ and Σ are not known	$W'm$ $W'SW$	$[\frac{(T^2-1)(T-n+1)}{T^2(T-n-1)}]$

From Table 1, portfolio variance of the third state is the largest and portfolio variance of the first state, or traditional mean-variance approach, has the smallest value. This can be interpreted as meaning that, as the uncertainty from estimation increases, the risk perceived by investors should increase. The application is that a rational investor will be more aware of risk. Therefore, given the same level of risk, a rational investor who takes into account the estimation risk will require a higher expected rate of return than one who forms a portfolio based on a traditional approach.

The admissible efficient set is a set of portfolio that yields the highest portfolio return given a level of risk, or a set of portfolio that has the lowest portfolio risk given a level of portfolio return. From Table 1, the portfolio risk of each state, $W'SW$, is the same, which can be interpreted as meaning that portfolio risk in each state is not affected by estimation risk. Only the constant term is multiplied to the portfolio risk, $W'SW$. An investor would be selecting the same admissible efficient set under the mean-variance analysis regardless of the state of analysis.

Let W'_a represents the vector of optimal proportion allocated to each asset under each state of analysis and let Ai be the vector of weights conditional on portfolio variance of each state.

Let K1 = $[1+\frac{1}{T}]$

K2 = $[\frac{(T^2-1)(T-n+1)}{T^2(T-n-1)}]$

Ai = Vector of weight conditional on portfolio variance, Ai, in each state.

Table 2: This table shows the admissible set under each state of analysis. There are three states in the analysis. The first state assumes that population or true parameters, μ and Σ, are known. The first state is typically assumed in traditional portfolio selection or mean variance analysis. The second state assumes that true variance-covariance Σ is known and μ is not known. The third state assumes that both population parameters μ and Σ are not known.

State of analysis	Vector A_i
Both μ and Σ are known	$A_1 = \{W \mid W\text{‘}SW = W_a\text{‘}SW_a\text{’}\}$
Σ is known and μ is not known	$A_2 = \{W \mid K_1\, W\text{‘}SW = K_1 W_a\text{‘}SW_a\text{’}\}$
Both μ and Σ are not known	$A_2 = \{W \mid K_2\, W\text{‘}SW = K_2 W_a\text{‘}SW_a\text{’}\}$

From Table 2, vectors of weight assigned to the optimal portfolio conditional on portfolio variance in three states are different due to the scale factors. Since K_1 and K_2 are constants, conditional weight assigned to each asset will not be affected by estimation risk. This can be substantiated by empirical evidence. When comparing the efficient frontier constructed from a traditional approach with that of the Bayesian approach, analytically, the Bayesian portfolio has higher risk for all levels of expected return. This implies that the Bayesian efficient frontier will always lies to the right of the traditional efficient frontier or lies below the traditional portfolio.

3.2 Effect of Estimation Risk on Investor Utility

Two major steps in portfolio theory are constructing an efficient frontier and determining an optimal portfolio according to investor preference. Constructing an efficient frontier is the first step in portfolio theory. Allocating one's wealth into two types of assets, namely risky and riskless assets, is the second step. An optimal portfolio is determined as a tangency between investor's utility and a portfolio efficient set. An investor always maximizes the expected utility of his or her end of period wealth. End of period wealth will be maximized based on expected return from investor portfolio decisions. This implies that to maximize end of period wealth, an investor decides on the weight of investment to maximize the utility of expected return on portfolio M = , where is the vector of future observed return.

$$\text{E[U(M)]} = \int_M U(M)\,p(M|\theta)\,dM \tag{18}$$

From the expected utility function above, investor utility function and the distribution of rate of return conditioned on a set of parameters must be known. A traditional portfolio formation approach assumes that true parameters can be estimated by sample parameters, (Ri), obtained from the historical rate of return on assets. An optimal portfolio under a traditional portfolio approach is obtained by the relationship shown below.

$$\max_w E_R[U(M)|\theta = \hat{\theta}(R_i)] \tag{19}$$

To incorporate estimation risk in portfolio theory, the predictive density function of asset returns should be used. Zellner and Chetty (1965) suggested that, in determining an optimal portfolio based on a Bayesian framework, by integrating out the unknown parameter from the predictive density, estimation risk is implicitly taken into account and the portfolio optimization problem can be described as the maximization of the unconditional expected utility.

$$\max_w E_\theta[E_{R|\theta}[U(M)|\theta]] \tag{20}$$

$$E_\theta[E_{R|\hat{\theta}}[U(M)|\theta] = \int_\theta \int_R U(M)\,p(M|\theta)\,dM \;\; p(\theta|R, I_0)\,d\theta \tag{21}$$

Where $p(\theta|R, I_o)$ is the posterior density function of θ, given observed return and prior information, I_o. We can state the posterior function as the product of density function of a likelihood observed return and prior belief of true parameters as follows.

$$= \ p(\theta|R, I_0) = f(R|\theta)p(\theta|I_0) \qquad (22)$$

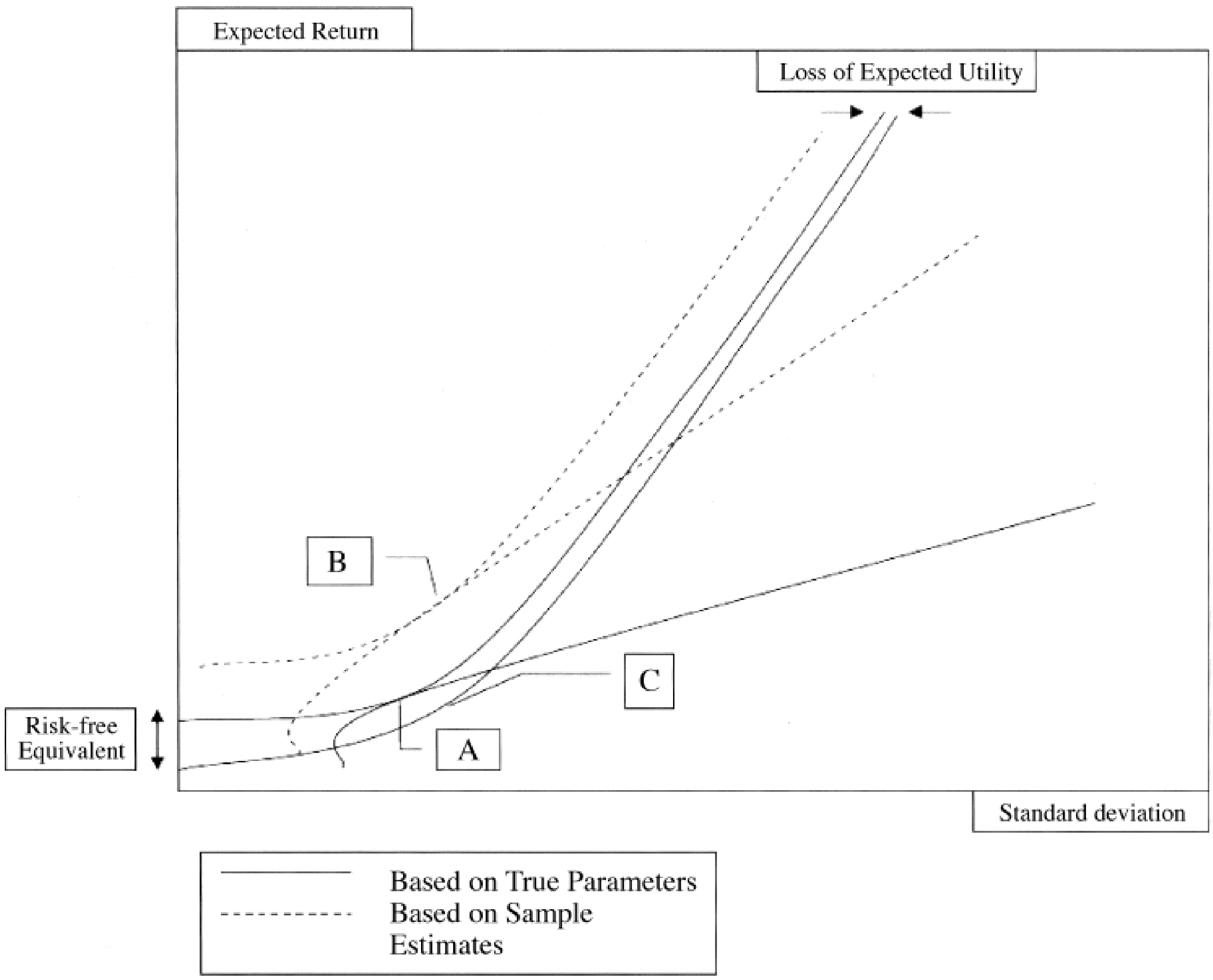
The expected utility optimization as a function of mean-variance can be stated as

$$E[U(M)] = Z(\mu_M, \sigma_M^2) \qquad (23)$$

where: μ_M, σ^2_M are portfolio return and variance, respectively.

As utility function depends on portfolio expected returns and variances and, as portfolio return is $w'R_i$,and portfolio variance is $w' \Sigma w$, optimal utility relies on the distribution moments, $\theta = (\mu, \Sigma)$. If the distribution moments of true parameters are known, the expected utility of each investor would be optimized. Let Z (μ^*_z, σ^{2*}_z) be an optimal expected utility.

Figure 1: Utility Loss Due To Estimation Risk



$$Z(\mu_Z^*, \sigma_Z^{2*}) = Z(w^*(\theta)|\mu, \Sigma) \equiv Z(w^{*\prime}\mu, w^{*\prime}\Sigma w^*) = Z_{\text{optimal}} \quad (24)$$

On the other hand, true parameters are not known. Sample parameters are used as estimates for true parameters in determining optimal portfolio choice. Expected utility of an optimal portfolio based on distribution moments of sample estimates is stated below.

$$Z(w(\hat{\theta}(R_i))|\mu, \Sigma) = Z(\hat{w}'\mu, \hat{w}'\Sigma\hat{w}) \quad (25)$$

It can be implied from Equations (24) and (25) that expected utility based on true parameter estimators is always higher than that based on sample estimates, Z ($\hat{w}'\mu, \hat{w}'\Sigma\hat{w}'$) $\leq Z_{optimal}$. This is because there is no estimation error in the portfolio optimization process when true parameters are treated as inputs. This implies that portfolio decisions based on sample moments cause loss in utility due to parameter uncertainty.

Following Brown (1976), loss in utility due to estimation risk is measured as

$$L(W^*, W) = \frac{Z_{optimal} - Z(\hat{w})}{|Z_{optimal}|} \quad (26)$$

Figure 1 depicts loss in utility due to estimation risk. If the true moments of asset return distribution are known, an optimal portfolio is located on solid line at point A, where the optimal weight is w*. If an efficient set is formed based on sample parameters, as suggested by a traditional portfolio approach, the dashed frontier, an optimal portfolio is located at point B, where the optimal weight is $\hat{w}$'. However, this choice $\hat{w}$ is suboptimal relative to the true parameters. This is because point C, with the same weight, $\hat{w}$, as that of traditional portfolio yields lower utility relative to utility at point A. The difference in value of expected utility between point A and point C is called utility loss due to estimation risk. Suboptimal portfolio decisions due to ignoring estimation risk will be explored empirically in Section 5.

4. Data

THERE are one hundred and sixty-six sectors listed on the New York Stock Exchange (NYSE). Information for these 166 index sectors was obtained from Data Stream. Periods in this study are weekly and monthly ranging from 1995 to 2004. There are 522 week-observations per sector and 120 month-observations per sector. The Internet sector (INTNT) was deleted due to incomplete data. This leaves 165 sectors for the screening process. Due to the large information set, two screening factors for selecting data were set in this study. The size and liquidity of each sector are used as screening factors. The proxy for size is the market capitalization of each sector and the proxy for liquidity is the turnover volume of each sector. The sectorial index used in this study is the sectorial index return adjusted dividend or Return Index (RI) from "Data Stream".

In the screening process, based on annual data from 1995 to 2004, the largest 30 sectors were selected based on size, market capitalization. Among those 30 sectors, the fifteen most liquid sectors by their turnover volume were selected as the data set. Market values and turnover volumes of the selected fifteen sectors represent 39.81 percent of the total market value and 54.64 percent of the total market turnover volumes. Table 3 shows the result of data screening as discussed.

As shown in Table 3, Technology Hardware and Equipment (INFOH) is the largest and is the most liquid sector. Semiconductors (SEMIC) is the lowest in market capitalization sector and Fixed Line Telecom (TELFL) is the least liquid sector. The Banking sector is ranked among the largest in market size but is ranked in the bottom 5 in turnover volume, whereas Semiconductors (SEMIC) and Telecom Equipment (TELEQ) are ranked in the bottom 5 of market capitalization but are ranked as the top 5 in turnover volume. This implies that a sector ranked as the largest in market capitalization may not be ranked as the most liquid sector, and vice versa.

Table 3: Summary of Screening Results.

Panel A of this table shows selected sectors based on two criteria and panel B shows ranked sectors by each criteria as top 5 and bottom 5.

Panel A: Result of data screening based on two criteria: size and liquidity.

	Market Value (in Million $)	Turnover Volumes (Thousands of Traded Share)
INFOH	919,200.36	81,637,018.40
TECHD	919,200.36	81,637,018.40
PHARM	808,065.43	20,151,565.13
BANKS	732,578.36	12,961,563.60
PHRMC	682,461.43	13,371,363.13
SFTCS	600,872.36	39,814,041.47
GNRET	502,692.29	17,124,408.27
TELCM	495,452.57	16,934,924.67
RTAIL	487,974.29	17,117,174.53
TELFL	429,564.00	12,309,664.53
SOFTW	394,993.36	27,241,120.13
COMPH	339,161.79	23,157,260.07
MEDIA	299,057.86	12,995,580.13
TELEQ	294,677.50	29,752,702.33
SEMIC	285,361.86	28,726,316.87

Panel B: This panel shows results from ranking selected sectors in top 5 and bottom 5 based on market value and turnover volume respectively.

Top 5		Bottom 5	
Market Value	MarketValue (in Million $)	Market Value	Market Value (in Million $)
INFOH	919,200.36	SOFTW	394,993.36
TECHD	919,200.36	COMPH	339,161.79
PHARM	808,065.43	MEDIA	299,057.86
BANKS	732,578.36	TELEQ	294,677.50
PHRMC	682,461.43	SEMIC	285,361.86
INFOH	81,637,018.40	TELCM	16,934,924.67
TECHD	81,637,018.40	PHRMC	13,371,363.13
SFTCS	39,814,041.47	MEDIA	12,995,580.13
TELEQ	29,752,702.33	BANKS	12,961,563.60
SEMIC	28,726,316.87	TELFL	12,309,664.53

5. Methodology and Empirical Evidence

THE empirical study in this section aims at comparing two types of efficient frontier, the traditional mean-variance efficient frontier and the Bayesian efficient frontier with selected conjugate prior distribution. The objective is to substantiate that, when estimation risk is taken into account, the admissible set of the efficient frontier will not be changed. As estimation risk is incorporated in portfolio formation, the Bayesian efficient set should always lie to the right of the traditional efficient set due to higher risk.

To substantiate the invariant admissible set when incorporating estimation risk in portfolio formation, two analyses of different period lengths are explored and compared. The first analysis is to construct and compare portfolio efficient sets and optimal weights for traditional and Bayesian portfolios for the long historical period. The second analysis is conducted over a one-year horizon by comparing efficient sets and optimal weights.

Optimal weights will be assigned based on a quadratic optimization approach. The objective function in portfolio formation is to minimize portfolio risk given a level of expected return. In this subsection, a Bayesian portfolio is formed under the assumption that both μ and Σ are not known and that joint distribution of $\tilde{\mu}$ and $\tilde{\Sigma}$ is normal-inverted Wishart family. Thus, the corresponding family of conjugate prior distributions is the normal-inverted Wishart family. Following analysis suggested by Winkler (1973), Bayesian portfolio return and portfolio variance will be estimated as follows:

Based on Equations (7) and (8), the objective functions in the portfolio optimization procedure of two types of efficient set are stated below.

Traditional approach:

$$\text{Obj.} \quad \text{Min } W'SW \qquad (27)$$

Bayesian Approach (state 3):

$$\text{Obj.} \quad \text{Min } \left[\frac{(T^2 - 1)(T - n + 1)}{T^2 (T - n\ 1\ 1}\right] W'SW \qquad (28)$$

where: T = number of observations

n = number of sectors

Efficient frontiers from both approaches, as shown in Figures 2 and 3, are of the same shape for both sub-periods. When comparing efficient sets between traditional and Bayesian approaches, both efficient sets do not cross each other and the Bayesian efficient set always lies to the right of the traditional portfolio. The empirical results support the analytical results stated by Brown (1979) and Klein and Bawa (1979). These results showed that estimation risk leads to different optimal portfolio choice leaving admissible efficient sets unaffected.

Barry (1974), Klein and Bawa (1979), and Brown (1979) have provided theoretical proof that estimation risk leads to different optimal portfolio choice while the admissible efficient sets are not affected. However, no empirical evidence has been proposed to support the claim and their analyses are based on the diffuse prior distribution case. In this study, it is found that the admissible efficient sets are the same and the optimal weights of the two approaches are not different, as shown in Table 4. However, the portfolio risk of the Bayesian portfolio is higher than that of the traditional portfolio at the same level of portfolio return. As a result, the Bayesian efficient portfolio always lies to the right of the traditional efficient portfolio as depicted in Figures 2 to 4. This implies that if investors form portfolios which are mean-variance efficient portfolios, portfolio risks are underestimated from a Bayesian portfolio perspective.

Empirical results shown in Table 5 indicate that annualized Bayesian portfolio risk is consistently higher than that of traditional portfolios. For monthly index return, annualized Bayesian portfolio risk is higher, ranging from 104 to 223 basis points in the first sub-period, and from 106 to 152 basis points in the second sub-period. For the weekly index return panel, annualized Bayesian portfolio risk is larger than that of a traditional portfolio in the range of 43 to 83 basis points in sub-period 1995-1999, and 51 to 72 basis points in sub-period 2000-2004.

It can be deduced from this section that portfolio decisions based on a traditional portfolio approach, ignoring estimation risk, would lead to a suboptimal portfolio. The empirical result in this section supports the analytical discussion in Section 3. When estimation risk is ignored, an investor would face a utility loss investment decision due to underestimated portfolio risk. Results

Table 4: Optimal weights of global minimum portfolio formed by the traditional approach are compared for two sub-periods, 1995-1999 and 2000-2004, using two index return bases, monthly index returns and weekly index returns with those of Bayesian approach.

Sub-Period	Return Basis	Strategy	BANKS	COMPH	GNRET	INFOH	MEDIA	PHARM	PHRMC	RTAIL	SEMIC	SFTC	SOFTW	TECHD	TELCM	TELEQ	TELFL	Port. Return	Port. Risk
1995-1999	Monthly	Trad.	0.00%	5.39%	7.12%	0.00%	40.23%	0.00%	30.27%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	16.99%	2.43%	3.90%
1995-1999	Monthly	Bayes	0.00%	5.39%	7.12%	0.00%	40.23%	0.00%	30.27%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	16.99%	2.43%	3.98%
1995-1999	Weekly	Trad.	0.00%	5.00%	3.18%	0.00%	43.22%	11.26%	0.00%	0.00%	1.65%	0.00%	0.00%	0.00%	0.00%	0.00%	35.70%	0.55%	2.03%
1995-1999	Weekly	Bayes	0.00%	5.00%	3.18%	0.00%	43.22%	11.26%	0.00%	0.00%	1.65%	0.00%	0.00%	0.00%	0.00%	0.00%	35.70%	0.55%	2.04%
2000-2004	Monthly	Trad.	29.51%	0.00%	14.87%	0.00%	0.00%	52.91%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	2.72%	0.16%	3.94%
2000-2004	Monthly	Bayes	29.51%	0.00%	14.87%	0.00%	0.00%	52.91%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	2.72%	0.16%	4.25%
2000-2004	Weekly	Trad.	16.87%	0.00%	4.24%	0.00%	18.38%	0.00%	43.53%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	16.98%	-0.07%	2.42%
2000-2004	Weekly	Bayes	16.87%	0.00%	4.24%	0.00%	18.38%	0.00%	43.53%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	16.98%	-0.07%	2.43%

Figure 2: This figure depicts two efficient frontiers, traditional mean-variance and Bayesian portfolio. Panel A shows two frontiers based on monthly index returns and Panel B shows two frontiers based on weekly index returns ranging from 1995 - 1999.

Figure 2 A: Comparing Two Efficient Frontiers Based on Monthly Index Return Ranging from 1995 - 1999

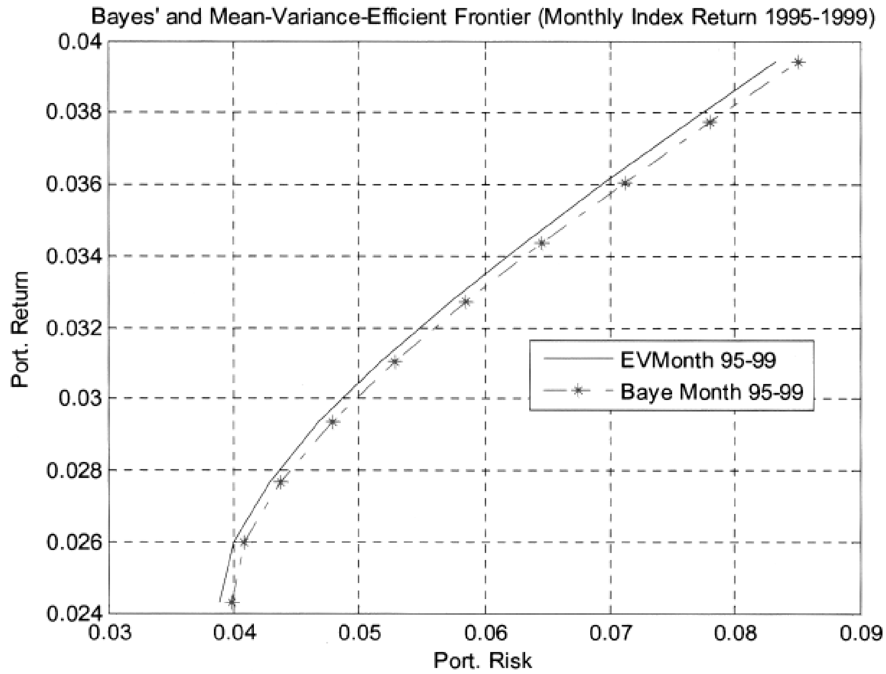


Figure 2 B: Comparing Two Efficient Frontiers Based on Weekly Index Return Ranging from 1995 - 1999

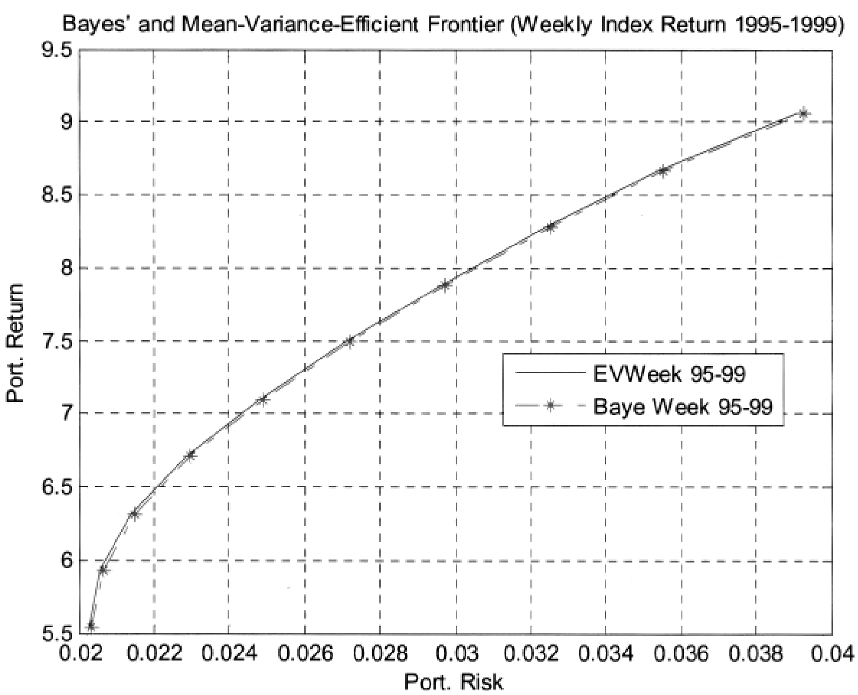


Figure 3: This figure depicts two efficient frontiers, traditional mean-variance and Bayesian portfolio. Panel A shows two frontiers based on monthly index returns and Panel B shows two frontiers based on weekly index returns ranging from 2000 - 2004.

Figure 3 A: Comparing Two Efficient Frontiers Based on Monthly Index Return Ranging from 2000 - 2004

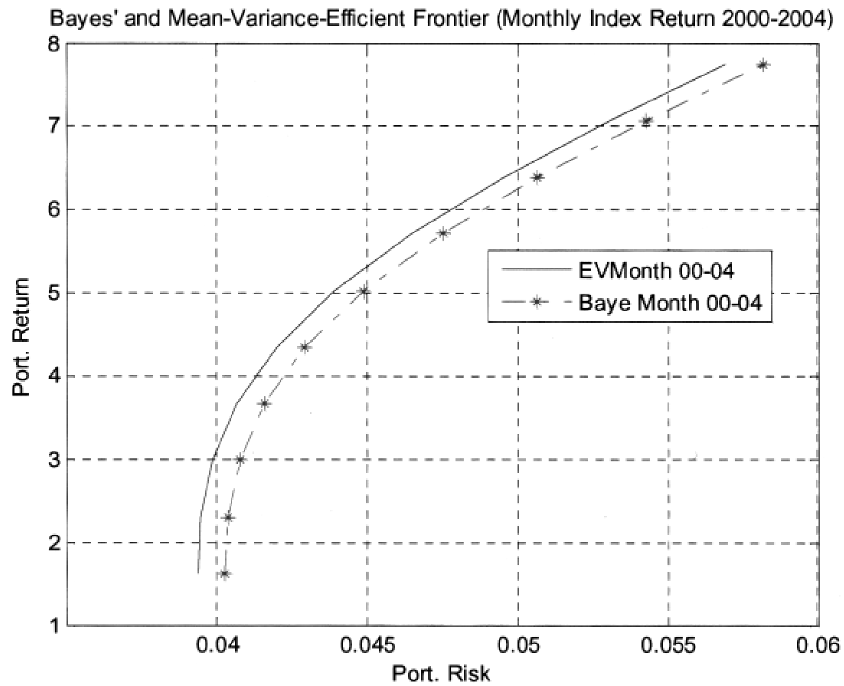


Figure 3 B: Comparing Two Efficient Frontiers Based on Weekly Index Return Ranging from 2000 - 2004

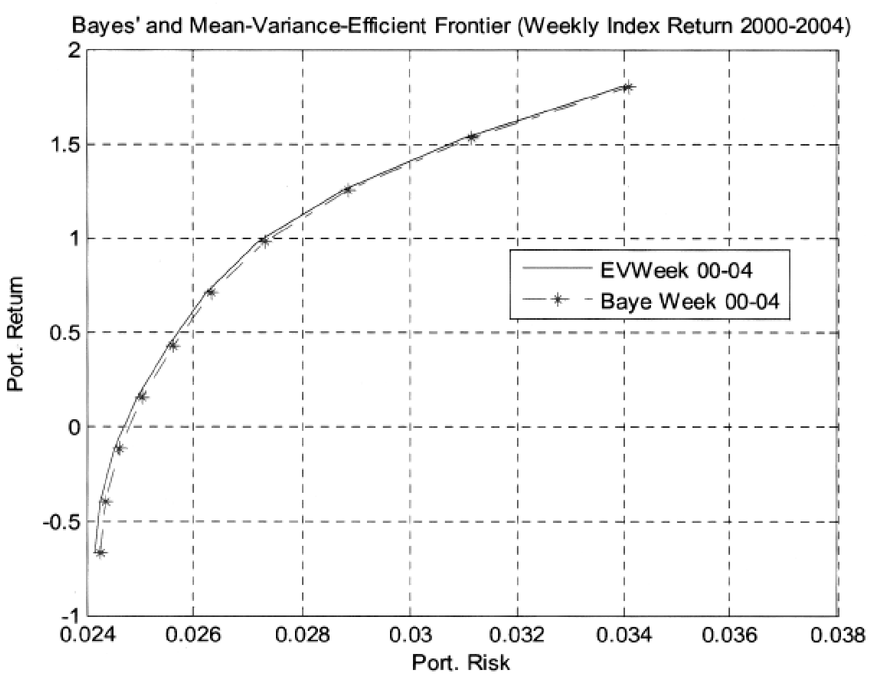


Figure 4: This figure depicts two efficient frontiers, traditional mean-variance and Bayesian portfolio based on annual analysis. In Panels A and B, two frontiers are based on monthly and weekly index returns of four sample years, which are 1998,1999, 2003, and 2004, representing two sub-periods, respectively.

Figure 4 A: Bayesian and Mean-Variance Efficient Frontier Based on Monthly Index Return

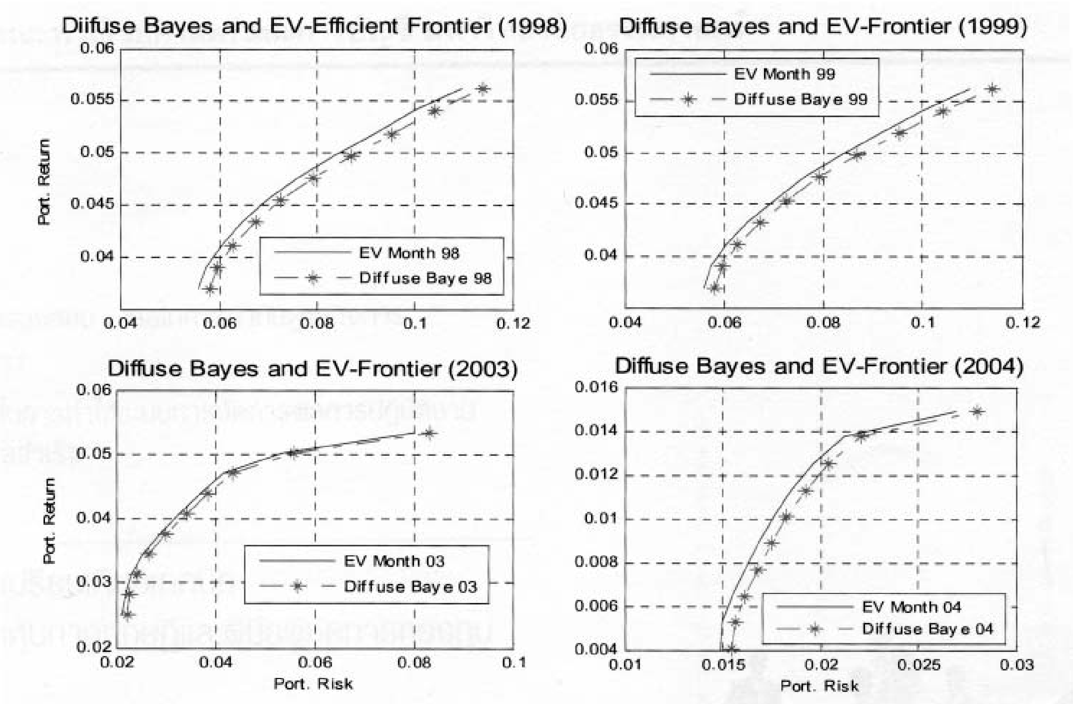


Figure 4 B: Bayesian and Mean-Variance Efficient Frontier Based on Weekly Index Return

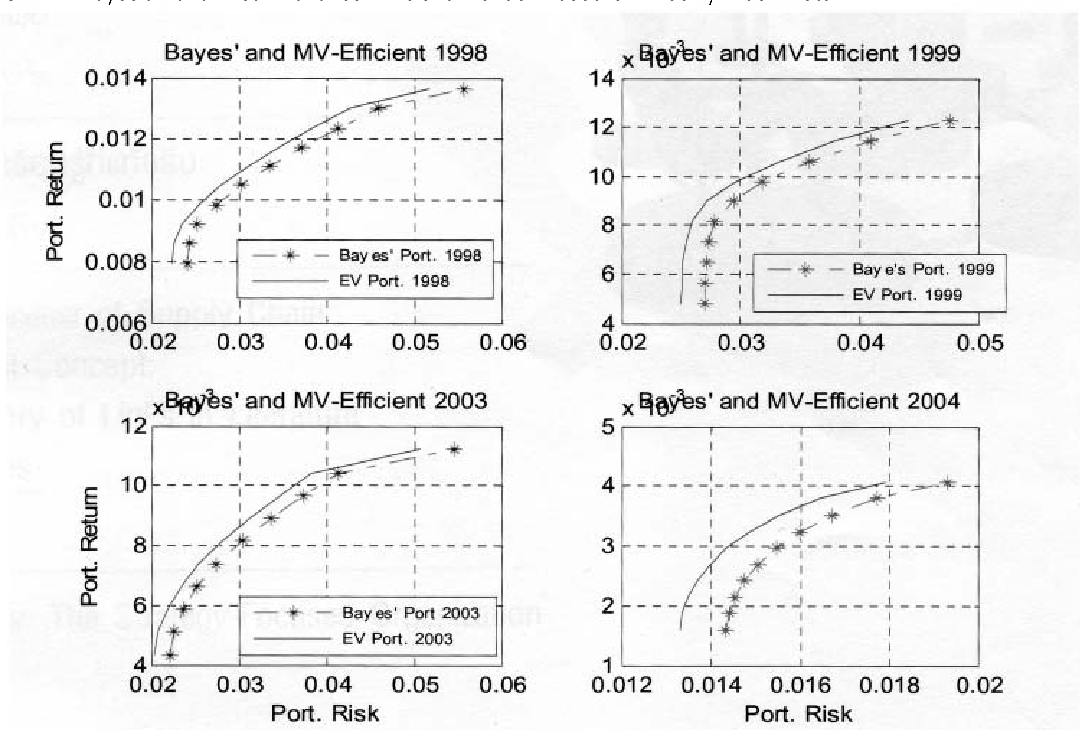


Table 5: This table shows the difference in portfolio risk between Bayesian and traditional portfolios in annualized terms measured in basis points. Annualized risk differences for weekly index returns and for monthly index returns are shown.

Sub-Period	Basis	Strategy		Port. 1	Port. 2	Port. 3	Port. 4	Port. 5	Port. 6	Port. 7	Port. 8	Port. 9	Port. 10
1995-1999	Monthly		Port. Return	0.16%	0.23%	0.30%	0.37%	0.43%	0.50%	0.57%	0.64%	0.71%	0.78%
		Traditional	Port. Risk	3.90%	4.00%	4.29%	4.68%	5.17%	5.72%	6.33%	6.97%	7.63%	8.33%
		Bayes	Port. Risk	3.98%	4.09%	4.38%	4.79%	5.29%	5.85%	6.47%	7.12%	7.80%	8.51%
			Annualized Risk Difference (Basis Points)	*104*	*107*	*115*	*126*	*139*	*153*	*170*	*187*	*205*	*223*
1995-1999	Weekly		Port. Return	0.55%	0.59%	0.63%	0.67%	0.71%	0.75%	0.79%	0.83%	0.87%	0.91%
		Traditional	Port. Risk	2.03%	2.06%	2.14%	2.29%	2.48%	2.71%	2.96%	3.24%	3.54%	3.91%
		Bayes	Port. Risk	2.04%	2.06%	2.15%	2.30%	2.49%	2.72%	2.98%	3.25%	3.56%	3.93%
			Annualized Risk Difference (Basis Points)	*43*	*43*	*45*	*48*	*52*	*57*	*63*	*69*	*75*	*83*
2000-2004	Monthly		Port. Return	0.16%	0.23%	0.30%	0.37%	0.43%	0.50%	0.57%	0.64%	0.71%	0.78%
		Traditional	Port. Risk	3.94%	3.95%	3.99%	4.07%	4.20%	4.39%	4.65%	4.96%	5.31%	5.69%
		Bayes	Port. Risk	4.03%	4.04%	4.08%	4.16%	4.29%	4.49%	4.75%	5.07%	5.42%	5.82%
			Annualized Risk Difference (Basis Points)	*106*	*106*	*107*	*109*	*112*	*118*	*125*	*133*	*142*	*152*
2000-2004	Weekly		Port. Return	-0.07%	-0.04%	-0.01%	0.02%	0.04%	0.07%	0.10%	0.13%	0.15%	0.18%
		Traditional	Port. Risk	2.42%	2.43%	2.45%	2.49%	2.55%	2.62%	2.72%	2.88%	3.11%	3.40%
		Bayes	Port. Risk	2.43%	2.44%	2.46%	2.50%	2.56%	2.63%	2.73%	2.89%	3.12%	3.41%
			Annualized Risk Difference (Basis Points)	*51*	*51*	*52*	*53*	*54*	*55*	*58*	*61*	*66*	*72*

discussed in this section are based on selected conjugate prior approach, the Normal-Wishart distribution, and the diffuse prior approach. For complete results regarding the effect of estimation risk on efficient set and optimal weights allocation, the empirical Bayesian portfolio is explored.

For each period, optimal weights are computed for each strategy. *Ex ante* portfolio return is computed for the following month. The first window, ranging from January 1992 to December 1996, is the base window for the optimal weights of the first period. *Ex ante* portfolio returns are computed and recorded for the next period, which is January 1997. Observed or ex post return in January 1997 for each sector is recorded based on the optimal weights from the *ex ante* portfolio. The same process is repeated for the period from January 1997 to December 2004, or 97 months of optimal weights. From these time series of *ex ante* and realized monthly returns and average portfolio risk, the Sharpe's ratios of those 97 portfolios are compared. A better portfolio strategy would yield a higher Sharpe's ratio and lower differences between *ex ante* and ex post average values.

Table 6 shows that every portfolio strategy always overestimates true parameter values. *Ex ante* average excess portfolio returns are higher than those of ex post averages for all strategies. Under *ex ante* average excess portfolio return, the Traditional Portfolio strategy is expected to have the largest average excess portfolio return compared with other strategies. *Ex ante* average excess portfolio return for traditional or mean-variance approach is 1.37 percent per month or 16.44 percent per annum. The lowest *ex ante* average excess portfolio returns are those of naïve and passive portfolio strategies (1.20 percent per month or 14.40 percent per annum). On the other hand, ex post average excess portfolio return for the Bayes-Stein portfolio is the highest at 0.88 percent per month or 10.56 percent per year and the lowest ex post average excess portfolio return is for the passive portfolio strategy (0.56 percent per month or 6.72 percent per annum). The largest deviation between ex

ante and ex post average excess portfolio return is for the passive portfolio approach (0.65 percent per month or 7.78 percent per year). Traditional portfolio approach has the second largest deviation (0.53 percent per month or 6.35 percent per year). The two lowest deviations between *ex ante* and ex post average excess portfolio return are for the Naïve and Bayes-Stein Approach (0.44 percent per month or 5.28 percent per year for Naïve portfolio and 0.45 percent per month or 5.40 percent per year for Bayes-Stein portfolio).

The last two columns in Table 6 report the Sharpe's Ratio of each strategy. Based on ex ante Sharpe's Ratio, the passive or value weighted approach produces the largest Sharpe's Ratio and the traditional or mean-variance approach has the lowest ratio. This may lead to the conclusion that the passive or value weighted approach yields a better performance on an ex ante basis. However, the Bayes-Stein shrinkage portfolio performs best on an ex post basis.

Table 6: Portfolio Performance of Alternative Estimation Meth

Strategy	Average Excess Portfolio Monthly Return		Portfolio Risk (Standard Deviation)**	Sharpe's Ratio	
	Ex ante Average	*Ex post* Average		*Ex ante* Average	*Ex post* Average
1. Naïve Portfolio					
Equal Weighted	1.20	0.76	5.63	0.2642	0.2087
2. Passive Portfolio					
Value Weighted	1.20	0.56	5.19	0.2723*	0.1809
3. Traditional Optimized Portfolio					
Mean-Variance	1.37	0.85	6.76	0.2542	0.1842
4. Optimized Portfolio: Shrinkage Estimator					
Bayes-Stein	1.33	0.88	6.43	0.2599	0.2290*

Note: *Ex ante* and *ex post* portfolio monthly return and risk are reported. Total sample periods range from January 1992 to December 2004. Out-of-sample periods range from January 1997 to December 2004.

* denotes the highest Sharpe's Ratio

* * portfolio risk is reported on one basis because the variance-covariance matrix is assumed to be known. This implies that the historical variance-covariance matrix is treated as true population parameter.

Sharpe's Ratios from the Bayes-Stein shrinkage approach is the largest at 0.2290. The value weighted approach yields the lowest Sharpe's Ratio of 0.1809 based on *ex post* average. Among optimized portfolio strategies, it can be concluded that the *ex post* performance of the Bayesian portfolio approach exceeded that of the traditional approach even though the Bayesian portfolio approach seems to perform worse than the Naïve and traditional approaches on an *ex ante* basis. Further studies on the estimation risk and portfolio selection could be conducted by applying shrinkage mean estimation incorporating asset pricing model in the informative prior function.

6. Conclusion

THIS study explores the effect of estimation risk on portfolio efficient sets based on U.S. sectorial index returns. Analytical discussion about the effect of estimation risk on efficient sets and empirical evidence supporting the analytical discussion are provided. Two points can be made from the analytical discussion. The first is that portfolio returns are the same in all three states, represented by the weighted average of sample average return, *W'm*. The second conclusion is that portfolio risk or portfolio variance when incorporating estimation risk differs from the traditional portfolio by a scale factor. Therefore, when estimation risk is taken into account, the admissible efficient set is not changed. The only effect from estimation risk is that the Bayesian efficient set will always lie to the right of the traditional efficient frontier due to higher risk from the estimation.

When estimation risk is incorporated into the portfolio decision process, it is shown that there exists loss in utility. Two findings from the empirical evidences from this study are as follows. Firstly, the optimal weights allocation of traditional and Bayesian portfolios are the same. However, the annualized Bayesian portfolio risk is larger than that of traditional portfolio by approximately 40 to 80 basis points on a weekly index return basis and by 100 to 220 basis points on a monthly index return

basis. Therefore, portfolio decisions based on a traditional approach, ignoring estimation risk, would lead to a suboptimal portfolio. Secondly, among four strategies analyzed in this study, Baye-Stein shrinkage portfolio outperforms other alternative by having largest ex post Sharpe's ratio and yields the lowest deviation between ex ante and ex post average excess portfolio return. The annualized average excess portfolio return from Bayes-Stein shrinkage portfolio is higher than those of traditional, passive, and naïve portfolio by 36, 384, and 144 basis points, respectively.

Appendix A

Derivation of Variance-Covariance Bayesian Portfolio Assuming Stationarity Variance-Covariance Matrix.

Let: S = known variance-covariance matrix

S* = estimated variance-covariance matrix

P = portfolio return based on true parameter mean, $W'\mu$

p = portfolio return based on estimated mean, $W'm$

To form an optimal portfolio incorporating estimation risk, the estimation error should be minimized. Estimation error is defined as [P - p]. Variance of the estimation error, represented by Var[P - p], is an appropriate measure and will be minimized.

$$\text{Var}[P - p] = \text{Var}[P] + \text{Var}[p] + 2\text{Cov}[P,p] \quad \text{(A.1)}$$

Since P and p are independent, thus, Cov{p,p] = 0 and we can simplify Equation (A.1) as

$$Var[P - p] = Var[P] + Var[p] \quad (A.2)$$

Since portfolio variance can be written in the following form

$$Var[P] = W'SW \quad (A.3)$$

$$Var[p] = W'SW$$

Rewrite Equation (A.2)

$$Var[P - p] = W'SW + W'SW$$

$$= W'(S + S^*)\, W \quad (A.4)$$

Since estimated variance-covariance, S*, is $(\frac{S}{T})$[4]. Equation (A.4) can be rewritten as

$$Var[P - p] = W'(S + \frac{S}{T})W = W'S(1 + \frac{1}{T})W = (1 + \frac{1}{T})W'SW$$

$$Var[P - p] = (1 + \frac{1}{T})S \quad (A.5)$$

APPENDIX B

Bayesian Variance-Covariance Matrix: Assuming Population Variance-Covariance is Known.

If R is a vector of asset return drawn from a normal distribution, $N(\theta, \sigma^2)$ where σ is assumed known, the likelihood function of parameter can be written as:

$$l(\theta|R) \propto (\frac{1}{\sqrt{2\pi\sigma}})^n \exp[-\frac{1}{2\sigma^2}\Sigma(R_i - \theta)^2] \quad (B.1)$$

where: $\propto$ = "proportion to"

$l(.)$ = likelihood function

Since variance is assumed known, we can incorporate the first term on the right of the function likelihood, , to the constant term. Thus, the likelihood function is rewritten in the following form:

$$l(\theta|\sigma, R) \propto \exp[-\frac{1}{2}(\frac{\theta - \bar{R}}{\sigma/\sqrt{T}})^2] \quad (B.2)$$

The result in Equation (B.2) is obtained by adding and subtracting sample mean, $\bar{R}$, to the last term in Equation (B.1), $\Sigma(R_i - \theta)^2$. After rearranging all terms, (B.1) can be rewritten as follows:

$$l(\theta|R) \propto (\frac{1}{\sqrt{2\pi\sigma}})^n \exp[-\frac{1}{2\sigma^2}\Sigma[(R_i - \bar{R}) - (\theta - \bar{R})]^2] \quad (B.3)$$

After distributing the square and summation operator, we have the following relation.

$$l(\theta|R) \propto (\frac{1}{\sqrt{2\pi\sigma}})^n \exp[-\frac{1}{2\sigma^2}\Sigma(R_i - \bar{R})^2 + T(\theta - \bar{R})^2] \quad (B.4)$$

Gather terms in Equation (B.4) and incorporate all constant terms to the proportional function, we shall have the relation as shown below.

$$l(\theta|R) \propto \exp[-\frac{T(\theta - \bar{R})^2}{2\sigma^2}] \quad (B.5)$$

$$l(\theta|R) \propto \exp[-(\frac{T}{2})(\frac{\theta - \bar{R}}{\sigma})^2] \quad (B.6)$$

Rearranging terms in Equation (B.6), we would have the final relationship as shown in Equation (B.2). Refer to the normal probability density function shown below

$$f(x) = (2\pi\sigma^2)^{-1/2} \exp[-\frac{1}{2}(\frac{R_i - \mu}{\sigma})^2] \quad (B.7)$$

We can infer, from Equations (B.2) and (B.7), that the sample mean, $\bar{R}$, represents population mean and $\frac{\sigma^2}{T}$ represents population variance-covariance matrix.

[4] See the proof of estimated variance-covariance under Bayesian approach when we assume population variance-covariance is know in Appendix C.

APPENDIX C

Lower and Upper Boundaries of Bayesian Adjusted Factor

From the third state of analysis, where both and are not known, the Bayesian adjusted factor is stated below:

$$[\frac{(T^2-1)(T-n+1)}{T^2(T-n-1)}] \quad \text{(C.1)}$$

where: T = number of observations

n = number of assets

T > n

From Equation (C.1), it is impossible for the scalar factor to take zero value because the number of observations and assets are non negative values. Four possible values of the Bayesian adjusted factor are negative, greater than zero but less than unity, unity, and larger than unity. Four scenarios are shown below.

(i) Bayesian adjusted factor is a negative value:

$$[\frac{(T^2-1)(T-n+1)}{T^2(T-n-1)}] < 0 \quad \text{(C.2.1)}$$

Since the numerator and denominator of Equation (C.1) are non negative values as the number of observations is greater than number of assets, then the Bayesian adjusted factor will be a non negative value.

$$(T^2-1)(T-n+1) > 0 \quad \text{(C.2.2)}$$

$$T^2(T-n-1) > 0 \quad \text{(C.2.3)}$$

(ii) Bayesian adjusted factor is greater than zero but less than unity:

$$0 \le [\frac{(T^2-1)(T-n+1)}{T^2(T-n-1)}] < 1 \quad \text{(C.3)}$$

The relationship can be restated as shown below.

$$(T^2-1)(T-n+1) < T^2(T-n-1) \quad \text{(C.4)}$$

Distribute and collect all terms in Equation (C.4) giving:

$$T^3 - T^2 n + T^2 - T + n - 1 < T^3 - T^2 n - T^2 \quad \text{(C.5)}$$

$$T(2T-1) < 1 - \mathrm{n} \quad \text{(C.6)}$$

From Equation (C.5), as the number of assets is a non negative value, the right-hand side of the equation is a negative value, from which it can be implied that the value on the left-hand side is also a negative value. Since the number of observations is also a non-negative value, it is impossible for $T(2T - 1)$ to be a negative value. Then, Equation (C.5) is not true and it can be concluded that the Bayesian adjusted factor is not a positive value between zero and unity.

(iii) Bayesian adjusted factor value is unity:

$$[\frac{(T^2-1)(T-n+1)}{T^2(T-n-1)}] = 1 \quad \text{(C.7)}$$

As stated in Equation (C.3), after distributing and collecting the terms, the relationship is stated below.

$$(T^2 - 1)(T - n + 1) = T^2 (T - n - 1) \qquad \text{(C.8)}$$

$$T(2T - 1) = 1 - \mathrm{n} \qquad \text{(C.9)}$$

The relationship in Equation (C.8) is not true because the left-hand side of the equation is a positive value whereas the right-hand side is a negative value. Hence, we can conclude that Bayesian adjusted factor value is not equal to unity.

(iv) Bayesian adjusted factor is larger than unity:

$$[\frac{(T^2 - 1)(T - n + 1)}{T^2 (T - n - 1)}] \geq 1 \qquad \text{(C.10)}$$

From Equation (C.10), the relationship can be stated as

$$T(2T - 1) > 1 - \mathrm{n} \qquad \text{(C.11)}$$

From the four scenarios, only the fourth scenario, or Equation (C.11), is compiled. It can be concluded that the Bayesian adjusted factor is a value larger than unity.

REFERENCES:

Adler, M., and Bernard, D. 1983, "International portfolio choice and corporation finance: A synthesis", *Journal of Finance,* vol. 38, no. 2, pp. 925-984.

Barberis, N. 2000, "Investing for the long run when returns are predictable", *Journal of Finance,* vol. 55, no. 1, pp. 225-264.

Barry, C. 1974, "Portfolio analysis under uncertain means, variances, and covariances", *Journal of Finance,* vol. 29, no. 2, pp. 515-522.

Barry, C., and Robert, W. 1976, "Nonstationarity and portfolio choice", *Journal of Financial and Quantitative Analysis,* vol. 11, no. 1, pp. 217-235.

Bawa, V., Stephen, B., and Klein R. 1979 *Studies in Bayesian Econometrics: Estimation Risk and Optimal Portfolio Choice,* North-Halland Publishing, New York.

Bawa, V. 1976, "Admissible portfolios for all individuals", *Journal of Finance,* vol. 31, no. 4, pp. 1169-1183.

Britten-Jones, M. 1999, "The sampling error in estimates of mean-variance efficient portfolio", *Journal of Finance,* vol. 54, no. 2, pp. 655-671.

Brown, S. 1979, "The effect of estimation risk on capital market equilibrium", *Journal of Financial and Quantitative Analysis,* vol. 14, no. 1, pp. 215-220.

Chen, S., and Brown, S., 1983, "Estimation risk and simple rules for optimal portfolio selection", *Journal of Finance,* vol. 38, no. 4, pp. 1087-1093.

Elton, E., Gruber, M., and Padberg, M., 1976, "Simple criteria for optimal portfolio selection", *Journal of Finance,* vol. 31, no. 5, pp. 1341-1357.

Frost, P., and Savarino, J., 1986, "An empirical Bayes approach to efficient portfolio selection", *Journal of Financial and Quantitative Analysis,* vol. 21, no. 2, pp. 293-305.

Greyserman, A., Jornes, D., and Strawderman, W., 2006, "Portfolio selection using Hierarchical Bayesian analysis and MCMC methods", *Journal of Banking and Finance,* vol. 30, no. 3, pp. 669-678.

Jorion, P., 1985, "International portfolio diversification with estimation risk", *Journal of Business,* vol. 58, no. 3, pp. 259-278.

Jorion, P., 1986, "Bayes-Stein estimation for portfolio analysis", *Journal of Financial and Quantitative Analysis,* vol. 21, no. 2, pp. 279-292.

Kalymon, B., 1971, "Estimation risk and portfolio selection model", *Journal of Financial and Quantitative Analysis,* vol. 6, no. 1, pp. 559-582.

Klein, R., and Bawa, V., 1976, "The effect of estimation risk on optimal portfolio choice", *Journal of Financial Economics,* vol. 3, no. 1, pp. 215-231.

Kroll, Y., Levy, H., and Markowitz, H., 1984, "Mean-variance versus direct utility maximization", *Journal of Finance,* vol. 39, no. 1, pp. 47 -61.

Markowitz, H., 1952, "Portfolio selection", *Journal of Finance,* vol. 7, no. 1, pp. 77-91.

Michaud, R., 1998, *Efficient Asset Management: A Practical Guide to Stock Portfolio Optimization and Asset Allocation,* Harvard Business School Press, Massachusetts.

Polson, N., and Tew, B., 2000, "Bayesian portfolio selection: An empirical analysis of the S&P 500 index 1970 -1996", *Journal of Business & Economic Statistics,* vol. 18, no. 1, pp. 164-173.

Winkler, R., 1973,"Bayesian models for forecasting future security prices", *Journal of Financial and Quantitative Analysis,* vol. 3, no. 2, 387-406.

Winkler, R., and Barry, C., 1975, "A Bayesian model for portfolio selection and revision", *Journal of Finance,* vol. 30, no. 1, 179-192.

Zellner, A., 1971, *An Introduction to Bayesian Inference in Econometrics,* John Wiley & Sons, New York.

Zellner, A., and Chetty, K., 1965, "Prediction and decision problems in regression models from the Bayesian point of view", *Journal of the American Statistical Association,* vol. 60, no. 2, pp. 608-616.